

NONMONOTONEITY OF PICARD PRINCIPLE

BY

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Dedicated to Professor Tadashi Kuroda on the occasion of his 60th birthday

ABSTRACT. Two nonnegative C^∞ functions $P(z)$ and $Q(z)$ on the punctured unit disk $0 < |z| \leq 1$ are constructed such that $Q(z) \leq P(z)$ and there exists only one Martin minimal boundary point for the equation $\Delta u = Pu$ over $z = 0$ and, nevertheless, there exist exactly two Martin minimal boundary points for the equation $\Delta u = Qu$ over $z = 0$.

The purpose of this paper is to show the existence of the following rather peculiar pair of two densities P and Q on the punctured unit disk Ω : $0 < |z| < 1$ with $0 \leq Q \leq P$ on Ω such that the Picard principle is valid for P and invalid for Q at the origin $z = 0$.

We start fixing terminologies before explaining our result more precisely. We will take the punctured sphere R : $0 < |z| \leq \infty$ as our base Riemann surface so that every topological notion will be considered relative to $R = \{0 < |z| \leq \infty\}$. Then the unit circle $|z| = 1$ is the relative boundary $\partial\Omega$ of the punctured disk Ω : $0 < |z| < 1$ and the origin $z = 0$ is the ideal boundary $\delta\Omega$ of Ω .

By a *density* P on Ω we mean a nonnegative locally Hölder continuous function $P(z)$ on the closure $\bar{\Omega} = \Omega \cup \partial\Omega$ so that P may or may not have a singularity at $z = 0$. With a density P on Ω we associate the class $PP(\Omega; \partial\Omega)$ of nonnegative real valued continuous functions u on $\bar{\Omega}$ vanishing on $\partial\Omega$ such that u satisfies the following selfadjoint elliptic equation of the second order:

$$(1) \quad \Delta u(z) = P(z)u(z)$$

on Ω , where Δ is the Laplacian $4\partial^2/\partial z\partial\bar{z}$. We also denote by $PP_1(\Omega; \partial\Omega)$ the subclass of $PP(\Omega; \partial\Omega)$ consisting of functions u with the normalization

$$(2) \quad -\int_{\partial\Omega} \frac{\partial}{\partial|z|} u(z) |dz| = 2\pi.$$

The cardinal number $\#(\text{ex. } PP_1(\Omega; \partial\Omega))$ of the set $\text{ex. } PP_1(\Omega; \partial\Omega)$ of extreme points of the convex set $PP_1(\Omega; \partial\Omega)$ is referred to as the *Picard dimension* of a density P at the ideal boundary $\delta\Omega$ of Ω , $\dim P$ in notation, i.e.

$$(3) \quad \dim P = \#(\text{ex. } PP_1(\Omega; \partial\Omega)).$$

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There exists a bijective correspondence $u \leftrightarrow \mu$ between $PP_1(\Omega; \partial\Omega)$ and the set of probability measures μ on $\text{ex. } PP_1(\Omega; \partial\Omega)$ such that

$$u = \int_{\text{ex. } PP_1(\Omega; \partial\Omega)} v \, d\mu(v)$$

as a consequence of the Choquet theorem (cf. e.g. [11]).

If $\dim P = 1$, then we say that the *Picard principle* is valid for P . This means that $PP_1(\Omega; \partial\Omega)$ consists of a single function u_0 so that for every nonnegative solution h of (1) on Ω there exist a nonnegative real number λ and a bounded solution b of (1) on $0 < |z| < r < 1$ such that $h = \lambda u_0 + b$. This property was first found by Picard in 1923 for the density $P \equiv 0$, i.e. for positive harmonic functions, with $u_0(z) = -\log|z|$, which is also known as the principle of positive singularities. The above formulation, as being of Picard dimension one, is due to Bouligand in 1931 and the study of the Picard principle for general densities was initiated by Brelot [1, 2].

We are interested in characterizing all densities P on Ω for which the Picard principle is valid. To a certain extent it is true that the Picard principle is valid for a density P at $\partial\Omega$ if the singular behavior of P at $\partial\Omega$ is not too wild. For example, if P is in $L^1(\Omega)$ [8] or $P(z) = O(|z|^{-2})$ as $z \rightarrow 0$ [4], then the Picard principle is valid for P . In view of these it may sound plausible that if the Picard principle is valid for a density P , then it is also valid for any density $Q \leq P$ since the singular behavior of Q at $\partial\Omega$ must be less wild than that of P . This is certainly the case [7, 5] for densities P and Q which are rotation free, i.e. $P(z) = P(|z|)$ and $Q(z) = Q(|z|)$. The purpose of this paper is to show, contrary to this intuition, that the above expectation is not in general correct. Namely we will prove the following:

THE MAIN THEOREM. *There exists a pair of densities P and Q on Ω such that $0 \leq Q \leq P$ on Ω and the Picard principle is valid for P and invalid for Q .*

The correspondence $P \mapsto \dim P$ is a mapping from the set \mathcal{D} of densities on Ω into the set of cardinal numbers. We have seen [9, 10] that the range $\dim \mathcal{D}$ contains the set \mathbf{N} of positive integers, the infinite countable cardinal number \aleph and the cardinal number \mathfrak{c} of the continuum so that $\dim \mathcal{D} \subset [1, \mathfrak{c}]$ in general and $\dim \mathcal{D} = [1, \mathfrak{c}]$ if the continuum hypothesis is postulated. We have asked [6] whether $P \mapsto \dim P$ is monotone (i.e. whether $P \leq Q$ implies $\dim P \leq \dim Q$). The above result shows that it is not monotone in general on \mathcal{D} although it is monotone [7] on \mathcal{D}_r , the set of rotation free densities.

The proof of the main theorem will be divided into three parts. In the first part, §1, two subregions of Ω with relative harmonic dimensions one and two, respectively, will be constructed. In §2 we associate what we call firmly associated densities with the above two regions. In both of these two parts, the Schwarz alternating method, or the linear operator method for principal functions in the modern terminology of Rodin-Sario [12], will be intensively used. The proof will be completed in the final very short §3.

1. Relative harmonic dimensions.

1. A sequence $\{\bar{Y}_n\}_1^\infty$ of closures \bar{Y}_n of Jordan regions Y_n in Ω will be referred to as a *\mathcal{Y} -sequence* in Ω if $\bar{Y}_n \cap \bar{Y}_m = \emptyset$ ($n \neq m$), so $W = W(\{\bar{Y}_n\}) = \Omega - \bigcup_1^\infty \bar{Y}_n$ is connected and $\{\bar{Y}_n\}$ converges to $\partial\Omega$: $z = 0$, i.e. there exist only a finite number of

\bar{Y}_n such that $\bar{Y}_n \cap \{\varepsilon \leq |z| < 1\} \neq \emptyset$ for any $\varepsilon > 0$. We denote by ∂D the relative boundary of a subregion D of the punctured sphere $R = \{0 < |z| \leq +\infty\}$ considered in R . We then consider the class $HP(W; \partial W)$ of nonnegative harmonic functions on W with vanishing boundary values on ∂W and the subclass $HP_1(W; \partial W)$ of $HP(W; \partial W)$ consisting of those functions u with the normalization (2). Similar to the Picard dimension we define the *relative harmonic dimension*, $\dim\{\bar{Y}_n\}$ in notation, of a \mathcal{U} -sequence $\{\bar{Y}_n\}$ at $\delta\Omega: z = 0$ by

$$\dim\{\bar{Y}_n\} = \#(\text{ex. } HP_1(W; \partial W)).$$

In subsection 6 we will give examples of \mathcal{U} -sequences $\{\bar{S}_n\}_1^\infty$ and $\{\bar{S}_{nj}\}_{j=1,2; n \geq 1}$ satisfying $\dim\{\bar{S}_n\} = 1$, $\dim\{\bar{S}_{nj}\} = 2$ and $\bigcup_1^\infty (\bar{S}_{n1} \cup \bar{S}_{n2}) \subset \bigcup_1^\infty \bar{S}_n$ after establishing auxiliary results in subsections 2–5.

2. We fix positive numbers a, δ, p, q and ρ with $0 < a - \delta < a + \delta < 1$, $0 < p - \rho < p < q < q + \rho < 2\pi$ and consider a subregion (see Figure 1)

$$A = A(b) = \{a - \delta < |z| < a + \delta, p - \rho < \arg z < p\}$$

$$\cup \{a < |z| < b, p \leq \arg z \leq q\}$$

$$\cup \{a - \delta < |z| < a + \delta, q < \arg z < q + \rho\}$$

of Ω for any number b with $a < b < a + \delta$. We also consider subsets F^-, F^+ of ∂A and subsets Γ^-, Γ^+ of A given by (see Figure 2)

$$F^- = \{|z| = a - \delta, p - \rho < \arg z < p\}$$

$$\cup \{a - \delta \leq |z| \leq a + \delta, \arg z = p - \rho\}$$

$$\cup \{|z| = a + \delta, p - \rho < \arg z < p\},$$

$$F^+ = \{|z| = a - \delta, q < \arg z < q + \rho\}$$

$$\cup \{a - \delta \leq |z| \leq a + \delta, \arg z = q + \rho\}$$

$$\cup \{|z| = a + \delta, q < \arg z < q + \rho\},$$

$$\Gamma^- = \Gamma^-(b) = \{a < |z| < b, \arg z = p\},$$

$$\Gamma^+ = \Gamma^+(b) = \{a < |z| < b, \arg z = q\}.$$

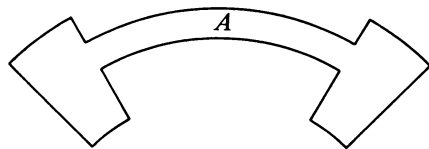


FIGURE 1

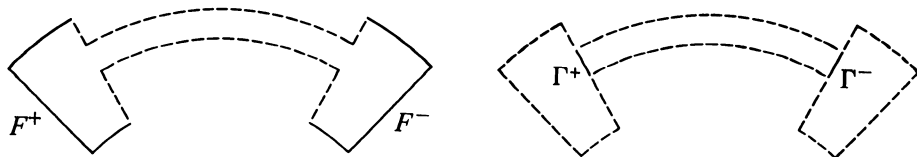


FIGURE 2



FIGURE 3

We denote by $\omega(\gamma; D, \cdot)$ the harmonic measure of a subset γ of ∂D considered on a subregion D of R . Then the family $\{u_b\}$ of harmonic measures

$$u_b(z) = \omega(F^- \cup F^+; A, z) \quad (z \in A)$$

converges to 0 on $\Gamma^- \cup \Gamma^+$ as $b \rightarrow a$:

LEMMA 1. $\lim_{b \rightarrow a} \sup_{\Gamma^- \cup \Gamma^+} u_b = 0$.

3. We prove Lemma 1. The harmonic measure u_b dominates the harmonic measure

$$w^-(z) = \omega(F^-; A \cap \{p - \rho < \arg z < p\}, z)$$

and is dominated by the harmonic measure

$$v_b^-(z) = \omega(F^- \cup \Gamma^-; A \cap \{p - \rho < \arg z < p\}, z).$$

Fix any positive number ε . Then there exist positive numbers δ_0, ρ_0 such that $\delta_0 < \delta, \rho_0 < \rho$ and $w^- < \varepsilon$ on (see Figure 3)

$$\begin{aligned} F_0^- = & \{|z| = a - \delta_0, p - \rho_0 < \arg z < p\} \\ & \cup \{a - \delta_0 \leq |z| \leq a + \delta_0, \arg z = p - \rho_0\} \\ & \cup \{|z| = a + \delta_0, p - \rho_0 < \arg z < p\}. \end{aligned}$$

The family $\{v_b^-\}$ converges decreasingly to w^- as $b \rightarrow a$ and hence by the Dini theorem v_b^- converges uniformly on every compact subset of

$$\bar{A} \cap \{p - \rho \leq \arg z \leq p\} - \{(a - \delta)e^{ip}, ae^{ip}, (a + \delta)e^{ip}\}.$$

Therefore there exists $b_0 = b_0(\varepsilon)$ with $a < b_0 < a + \delta_0$ such that $v_b^- - w^- < \varepsilon$ on F_0^- for every b with $a < b \leq b_0$. Then we have $u_b < v_b^- < 2\varepsilon$ on F_0^- and similarly $u_b < v_b^+ < 2\varepsilon$ on

$$\begin{aligned} F_0^+ = & \{|z| = a - \delta_0, q < \arg z < q + \rho_0\} \\ & \cup \{a - \delta_0 \leq |z| \leq a + \delta_0, \arg z = q + \rho_0\} \\ & \cup \{|z| = a + \delta_0, q < \arg z < q + \rho_0\}, \end{aligned}$$

where

$$v_b^+(z) = \omega(F^+ \cup \Gamma^+; A \cap \{q < \arg z < q + \rho\}, z).$$

By the maximum principle, $u_b < 2\varepsilon$ is valid on a region (see Figure 4)

$$\begin{aligned} & \{a - \delta_0 < |z| < a + \delta_0, p - \rho_0 < \arg z < p\} \\ & \cup \{a < |z| < b, p \leq \arg z \leq q\} \\ & \cup \{a - \delta_0 < |z| < a + \delta_0, q < \arg z < q + \rho_0\} \end{aligned}$$

including $\Gamma^- \cup \Gamma^+$. \square

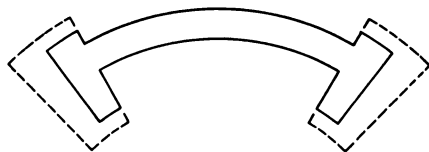


FIGURE 4

4. Let D be a subregion of R , γ a subset of ∂D consisting of a finite number of disjoint analytic arcs such that $\overline{\partial D - \gamma} \cap (\gamma - e(\gamma)) = \emptyset$ for the set $e(\gamma)$ of end points of each arc in γ . The Harnack constant $k = k(F)$ of a closed subset F of D such that \bar{F} is compact and contained in $D \cup (\gamma - e(\gamma))$ is given by

$$\begin{aligned} k &= k(F) = k(F; D, \gamma, a) \\ &= \sup \left\{ \frac{u(z)}{u(a)}; z \in F, u \in HP(D; \gamma) - \{0\} \right\}, \end{aligned}$$

where $a \in D$ and $HP(D; \gamma)$ is the set of nonnegative harmonic functions on D with vanishing boundary values on γ . Then we have the following

LEMMA 2. $k(F) < \infty$.

This is a special case of the so-called Carleson lemma but in our case of two dimension the existence of the conformal mapping reduces the lemma to a simple estimation of the Poisson kernel, which we describe in the sequel for the sake of completeness.

5. We assume that F contains a since $k(F) \leq k(F \cup \{a\})$. Let $\gamma_1, \dots, \gamma_s$ be disjoint analytic arcs with $\gamma = \gamma_1 \cup \dots \cup \gamma_s$. Then we divide F into a compact subset F_0 of D and closed subsets F_1, \dots, F_s of D satisfying $F = F_0 \cup F_1 \cup \dots \cup F_s$, $a \in F_0 \cap F_1 \cap \dots \cap F_s$, $\bar{F}_j \subset D \cup (\gamma_j - e(\gamma_j))$ ($j = 1, \dots, s$), and F_j is contained in a simply connected subregion of R . Further we take an analytic arc Γ_j such that $\Gamma_j \cup \gamma_j$ encloses a simply connected subregion D_j of D containing F_j . Observe the facts that $k(F) \leq \max_{0 \leq j \leq s} k(F_j)$, $k(F_0) < \infty$, $k(F_j) \leq k(F_j; D_j, \gamma_j, a)$, and D_j may be viewed as the unit disk by using a conformal mapping. Now it is sufficient to prove Lemma 2 in the case that $D = \{|z| < 1\}$, γ is a single circular arc, $a = 0$ and a is in F .

Let u be any positive function in $HP(D; \gamma)$ and μ be the Poisson-Herglotz (Martin) representing measure of u :

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left(\frac{\xi + z}{\xi - z} \right) d\mu(\xi).$$

The support of μ is contained in $\Gamma = \{|z| = 1\} - (\gamma - e(\gamma))$ which has a positive distance d from F . Then the Poisson kernel satisfies that

$$\operatorname{Re} \left(\frac{\xi + z}{\xi - z} \right) \leq \frac{|\xi + z|}{|\xi - z|} \leq \frac{2}{d} \quad (\xi \in \Gamma, z \in F).$$

Therefore we have

$$u(z) \leq \frac{1}{2\pi} \int_{\Gamma} \frac{2}{d} d\mu(\xi) = \frac{2}{d} u(0) \quad (z \in F)$$

so that $k(F) \leq 2/d$. \square

6. In this subsection we give two \mathcal{A} -sequences $\{\bar{S}_n\}_1^\infty$ and $\{\bar{S}_{nj}\}_{j=1,2;n \geq 1}$ with $\bigcup_1^\infty (\bar{S}_{n1} \cup \bar{S}_{n2}) \subset \bigcup_1^\infty \bar{S}_n$ and prove in subsections 7–9 that $\dim\{\bar{S}_n\} = 1$, $\dim\{\bar{S}_{nj}\} = 2$. Fix a sequence $\{a_n\}_1^\infty$ in $(0, 1)$ with $a_n > a_{n+1}$ and $\lim_n a_n = 0$. Fix positive numbers p_1, q_1, p_2, q_2 with $0 < p_1 < q_1 < p_2 < q_2 < 2\pi$. Then we consider \mathcal{A} -sequences $\{\bar{S}_n\}_1^\infty, \{\bar{S}_{nj}\}_{j=1,2;n \geq 1}$ for any sequence $\{b_n\}_1^\infty$ in $(0, 1)$ with $a_n > b_n > a_{n+1}$ given by (see Figure 5)

$$S_n = S_n(\{b_n\}) = \{b_n < |z| < a_n, p_1 < \arg z < q_2\},$$

$$S_{nj} = S_{nj}(\{b_n\}) = \{b_n < |z| < a_n, p_j < \arg z < q_j\} \quad (j = 1, 2; n = 1, 2, \dots).$$

We also consider subregions (see Figures 6 and 7)

$$\Omega_1 = \Omega - \{0 < |z| \leq a_1, p_1 \leq \arg z \leq q_2\},$$

$$\Omega_2 = \Omega - \bigcup_{j=1}^2 \{0 < |z| \leq a_1, p_j \leq \arg z \leq q_j\},$$

$$W_1 = W_1(\{b_n\}) = W(\{\bar{S}_n\}) = \Omega - \bigcup_{n=1}^\infty \bar{S}_n,$$

$$W_2 = W_2(\{b_n\}) = W(\{\bar{S}_{nj}\}) = \Omega - \bigcup_{n=1}^\infty (\bar{S}_{n1} \cup \bar{S}_{n2})$$

and mappings

$$T_\nu u = u - H_\nu^{\Omega_\nu} \quad (u \in HP(W_\nu; \partial W_\nu); \nu = 1, 2)$$

of $HP(W_\nu; \partial W_\nu)$ to $HP(\Omega_\nu; \partial \Omega_\nu)$, where H_f^D is defined as follows (cf. e.g. [3]). Let D be a nonvoid open subset of R not necessarily relatively compact such that each point of ∂D is contained in a nondegenerate continuum contained in ∂D . Let f be a continuous function on ∂D such that there exists a nonnegative superharmonic function s on D whose lower limit boundary values exceed $|f|$ on ∂D . Firstly, in case $f \geq 0$ on ∂D , we define H_f^D to be the lower envelope of the family of nonnegative superharmonic functions s on D with lower limit boundary values of $s \geq f$ on ∂D . For a general f on ∂D we define

$$H_f^D = H_{\max(f, 0)}^D - H_{\max(-f, 0)}^D.$$

Then H_f^D is harmonic on D and has boundary values f on ∂D . In case $f \geq 0$, H_f^D is the least nonnegative harmonic function on D with boundary values f on ∂D . It is easy to see that the mappings T_ν are order preserving (i.e. $u_1 \leq u_2$ implies $T_\nu u_1 \leq T_\nu u_2$), positively homogeneous (i.e. $T_\nu(\lambda u) = \lambda T_\nu u$ for nonnegative numbers λ), and additive (i.e. $T_\nu(u_1 + u_2) = T_\nu u_1 + T_\nu u_2$). If we choose the sequence $\{b_n\}$ so as to make the sequence $\{b_n - a_{n+1}\}_1^\infty$ converge to zero rapidly enough, as described in the sequel, then we can show that the mappings T_1, T_2 are bijective.

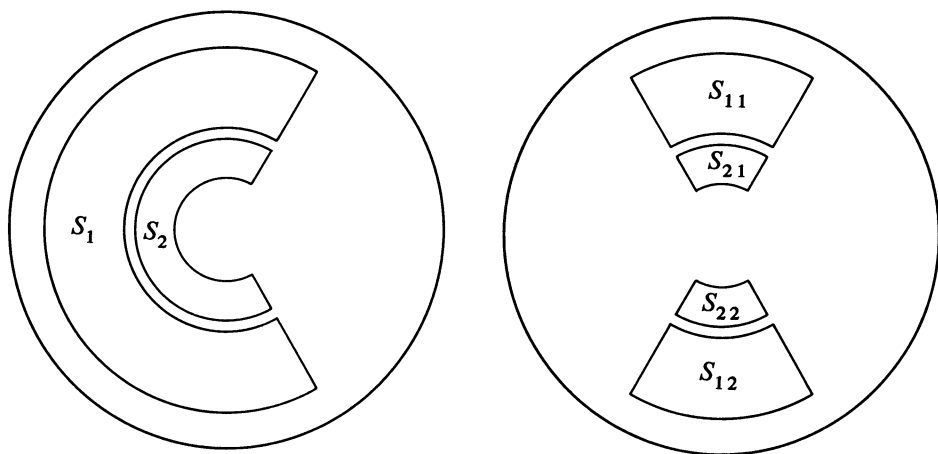


FIGURE 5

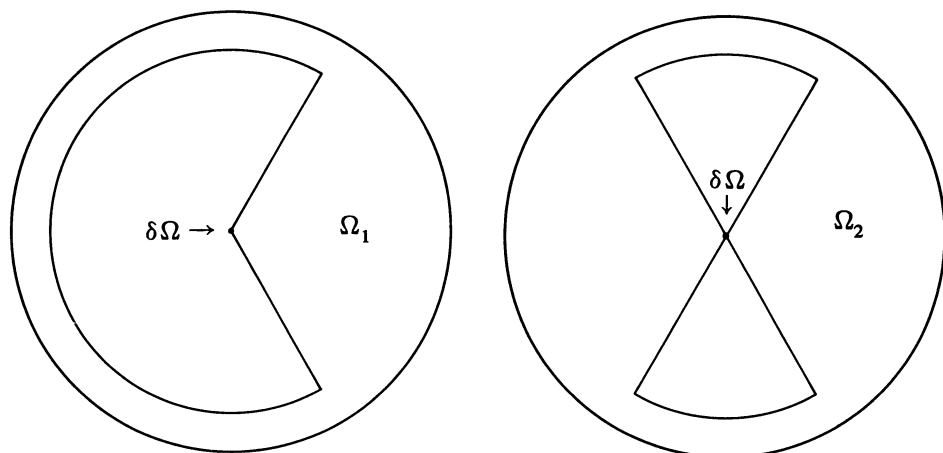


FIGURE 6

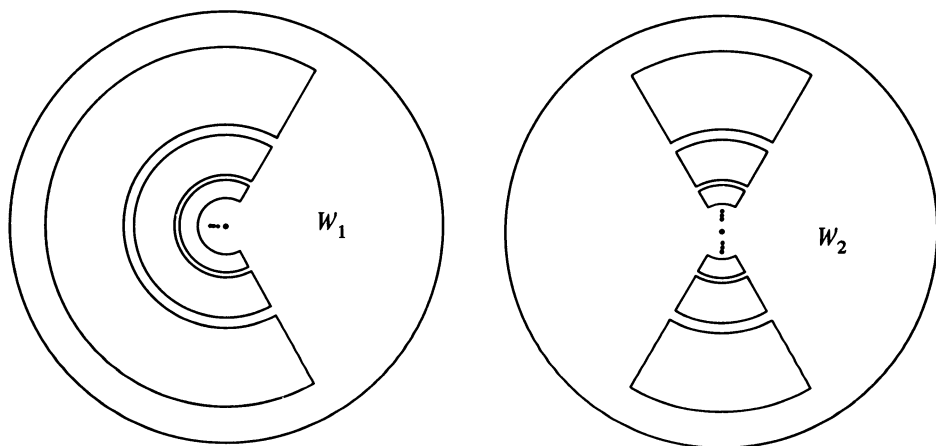


FIGURE 7

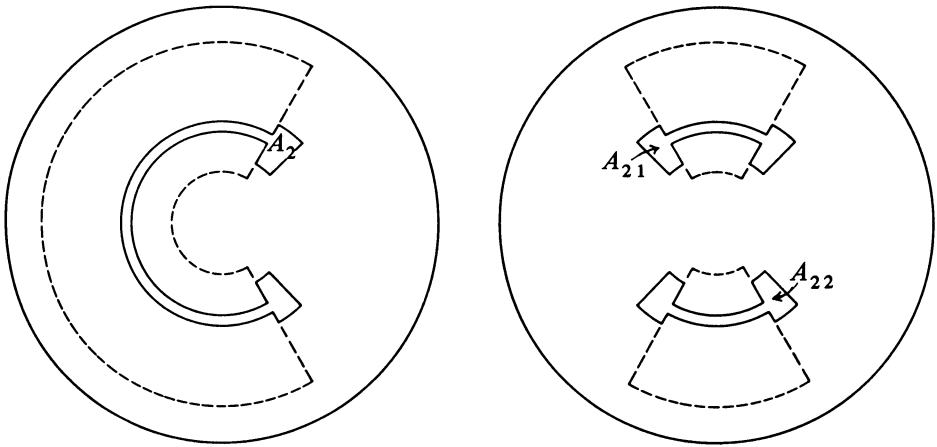


FIGURE 8

We fix a positive number ρ with

$$\rho < \min(p_1, (p_2 - q_1)/2, 2\pi - q_2)$$

and a sequence $\{\delta_n\}_2^\infty$ of positive numbers δ_n with

$$\delta_n < \min((a_{n-1} - a_n)/2, (a_n - a_{n+1})/2).$$

Then the first property which $\{b_n\}$ has to satisfy is

$$(4) \quad b_n < a_{n+1} + \delta_{n+1}/2 \quad (n = 1, 2, \dots).$$

For the sequence $\{b_n\}$ with this property we consider subregions (see Figure 8)

$$\begin{aligned} A_n &= A_n(\{b_n\}) \\ &= W_1 \cap \{a_n - \delta_n < |z| < a_n + \delta_n, p_1 - \rho < \arg z < q_2 + \rho\}, \\ A_{nj} &= A_{nj}(\{b_n\}) \\ &= W_2 \cap \{a_n - \delta_n < |z| < a_n + \delta_n, p_j - \rho < \arg z < q_j + \rho\} \\ &\quad (j = 1, 2; n = 2, 3, \dots) \end{aligned}$$

of W_1 , W_2 , respectively and closed subsets (see Figure 9)

$$\begin{aligned} F_n^- &= \partial A_n \cap \{p_1 - \rho \leq \arg z < p_1\}, \\ F_n^+ &= \partial A_n \cap \{q_2 < \arg z \leq q_2 + \rho\}, \\ F_{nj}^- &= \partial A_{nj} \cap \{p_j - \rho \leq \arg z < p_j\}, \\ F_{nj}^+ &= \partial A_{nj} \cap \{q_j < \arg z \leq q_j + \rho\} \quad (j = 1, 2; n = 2, 3, \dots) \end{aligned}$$

of Ω_1 or Ω_2 .

We also consider closed subsets (see Figure 10)

$$\begin{aligned} \Gamma_n^- &= \Gamma_n^-(\{b_n\}) = \{a_n < |z| < b_{n-1}, \arg z = p_1\}, \\ \Gamma_n^+ &= \Gamma_n^+(\{b_n\}) = \{a_n < |z| < b_{n-1}, \arg z = q_2\}, \\ \Gamma_{nj}^- &= \Gamma_{nj}^-(\{b_n\}) = \{a_n < |z| < b_{n-1}, \arg z = p_j\}, \\ \Gamma_{nj}^+ &= \Gamma_{nj}^+(\{b_n\}) = \{a_n < |z| < b_{n-1}, \arg z = q_j\} \end{aligned}$$

of A_n or A_{nj} and subsets (see Figure 11)

$$\begin{aligned}\gamma_n^- &= \{a_n - 5\delta_n/4 \leq |z| \leq a_n - 3\delta_n/4, \arg z = p_1\} \\ &\cup \{a_n + 3\delta_n/4 \leq |z| \leq a_n + 5\delta_n/4, \arg z = p_1\}, \\ \gamma_n^+ &= \{a_n - 5\delta_n/4 \leq |z| \leq a_n - 3\delta_n/4, \arg z = q_2\} \\ &\cup \{a_n + 3\delta_n/4 \leq |z| \leq a_n + 5\delta_n/4, \arg z = q_2\}, \\ \gamma_{nj}^- &= \{a_n - 5\delta_n/4 \leq |z| \leq a_n - 3\delta_n/4, \arg z = p_j\} \\ &\cup \{a_n + 3\delta_n/4 \leq |z| \leq a_n + 5\delta_n/4, \arg z = p_j\}, \\ \gamma_{nj}^+ &= \{a_n - 5\delta_n/4 \leq |z| \leq a_n - 3\delta_n/4, \arg z = q_j\} \\ &\cup \{a_n + 3\delta_n/4 \leq |z| \leq a_n + 5\delta_n/4, \arg z = q_j\}\end{aligned}$$

$$(j = 1, 2; n = 2, 3, \dots)$$

of $\partial\Omega_1$ or $\partial\Omega_2$.

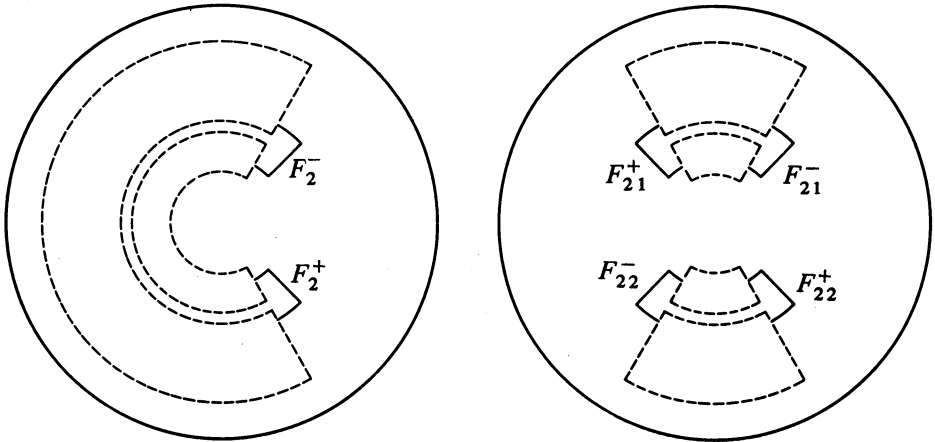


FIGURE 9

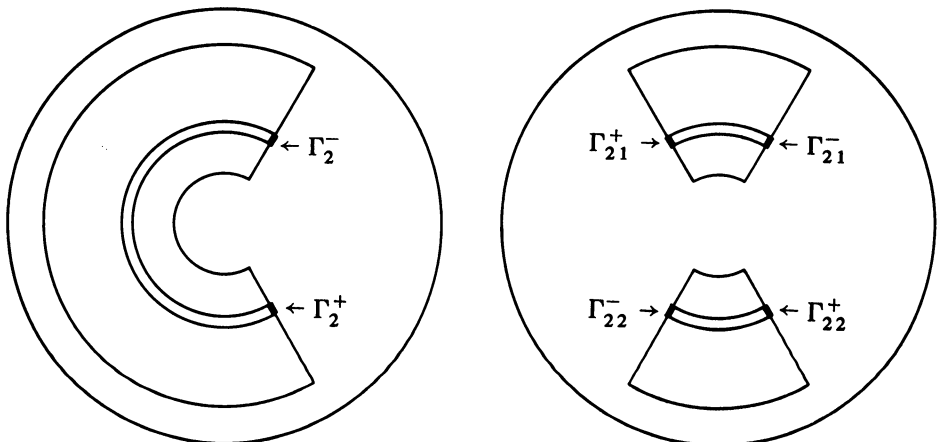


FIGURE 10

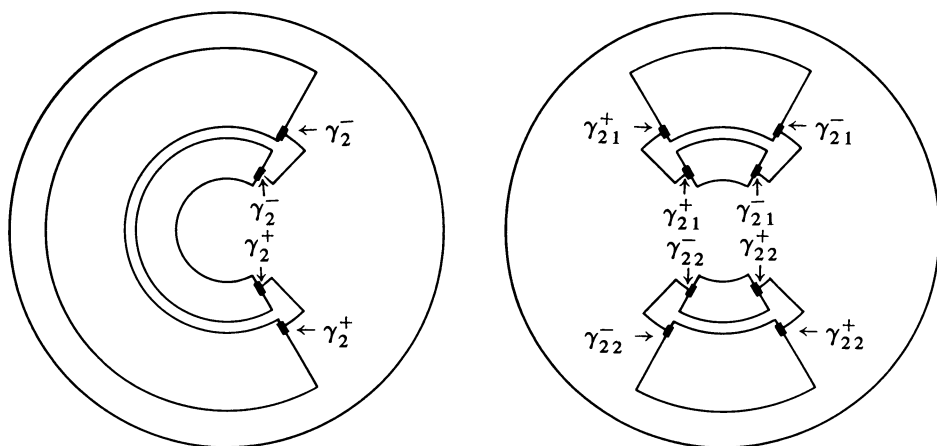


FIGURE 11

Then Harnack constants

$$\begin{aligned} k_n^- &= k(F_n^-; \Omega_1, \gamma_n^-, 1/2), & k_n^+ &= k(F_n^+; \Omega_1, \gamma_n^+, 1/2), \\ k_{nj}^- &= k(F_{nj}^-; \Omega_2, \gamma_{nj}^-, 1/2), & k_{nj}^+ &= k(F_{nj}^+; \Omega_2, \gamma_{nj}^+, 1/2) \end{aligned}$$

are finite by Lemma 2 and independent of $\{b_n\}$. Set

$$k_n = \max(k_n^-, k_n^+, k_{n1}^-, k_{n1}^+, k_{n2}^-, k_{n2}^+).$$

In view of Lemma 1 we can choose the sequence $\{b_n\}$ so as to satisfy

$$(5) \quad \begin{cases} \omega(F_n^- \cup F_n^+; A_n, z) \leq 1/2k_n & (z \in \Gamma_n^- \cup \Gamma_n^+), \\ \omega(F_{nj}^- \cup F_{nj}^+; A_{nj}, z) \leq 1/2k_{nj} & (z \in \Gamma_{nj}^- \cup \Gamma_{nj}^+) \end{cases}$$

for every $j = 1, 2$ and $n = 2, 3, \dots$. Now the second property which $\{b_n\}$ has to satisfy is (5). Then we will prove in subsections 7–8 the following

THEOREM 1. *If the sequence $\{b_n\}_1^\infty$ satisfies (4) and (5), then the mappings T_1 and T_2 are bijective.*

7. First we prove that T_2 is *injective*. The similar argument will also prove the injectivity of T_1 . Let u_1, u_2 be any functions in $HP(W_2; \partial W_2)$. Then u_1, u_2 satisfy

$$u_1 \leq k_n u_1(1/2), \quad u_2 \leq k_n u_2(1/2)$$

on $F_{nj}^- \cup F_{nj}^+$ ($j = 1, 2; n = 2, 3, \dots$). Therefore, by (5), u_1, u_2 satisfy

$$u_1 \leq u_1(1/2)/2, \quad u_2 \leq u_2(1/2)/2$$

on $\Gamma_2 = \bigcup_{n=2}^\infty \bigcup_{j=1}^2 (\Gamma_{nj}^- \cup \Gamma_{nj}^+)$. On the other hand u_1, u_2 vanish on $\partial\Omega_2 - \Gamma_2$ so that $u_1 - u_2$ is bounded on $\partial\Omega_2$. Assume $T_2 u_1 = T_2 u_2$. Since $u_1 - u_2 = H_{u_1 - u_2}^{\Omega_2}$ on Ω_2 , $u_1 - u_2$ is bounded on Ω_2 and hence on W_2 . Further $u_1 - u_2$ vanishes on ∂W_2 . Then we have $u_1 - u_2 \equiv 0$.

8. Next we prove that T_2 is *surjective*. The similar argument will also prove the surjectivity of T_1 . We set

$$F_2 = \bigcup_{n=2}^{\infty} \bigcup_{j=1}^2 (F_{nj}^- \cup F_{nj}^+), \quad \Gamma_2 = \bigcup_{n=2}^{\infty} \bigcup_{j=1}^2 (\Gamma_{nj}^- \cup \Gamma_{nj}^+).$$

Consider a bounded operator K_2 of $C(F_2)$ to $C(\Gamma_2)$ defined by

$$K_2\phi(z) = H_{\phi}^{A_{nj}}(z) \quad (\phi \in C(F_2), z \in \Gamma_{nj}^- \cup \Gamma_{nj}^+; j = 1, 2; n = 2, 3, \dots),$$

where $C(E)$ denotes the set of continuous functions on a subset E of R and for each $\phi \in C(F_2)$ we define $\phi = 0$ on $\bigcup_{n=2}^{\infty} (\partial A_{n1} \cup \partial A_{n2}) - F_2$. Also consider a bounded operator L_2 of $CB(\Gamma_2)$ to $CB(F_2)$ defined by

$$L_2\psi(z) = H_{\psi}^{\Omega_2}(z) \quad (\psi \in CB(\Gamma_2), z \in F_2),$$

where $CB(E)$ denotes the set of bounded continuous functions on a subset E of R and for each $\psi \in CB(\Gamma_2)$ we define $\psi = 0$ on $\partial\Omega_2 - \Gamma_2$. Let v be any function in $HP(\Omega_2; \partial\Omega_2)$. Thus we can construct a function u in $HP(W_2; \partial W_2)$ with $T_2u = v$ by using the bounded operator $M_2 = L_2 \circ K_2$ of a subspace of $C(F_2)$ to $CB(F_2)$. Since v is dominated by the constant $k_nv(1/2)$ on $\bigcup_{j=1}^2 (F_{nj}^- \cup F_{nj}^+)$ ($n = 2, 3, \dots$), we have, by (5), $K_2v \leq v(1/2)/2$ on Γ_2 . Then the harmonic function M_2v on Ω_2 is dominated by the constant $v(1/2)/2$ and hence by induction M_2^mv is dominated by the constant $v(1/2)/2^m$ ($m = 1, 2, \dots$). Therefore there exists a nonnegative function $\phi_v = \sum_{m=0}^{\infty} M_2^mv$ on F_2 with $(I - M_2)\phi_v = v$, where I is the identity operator of $C(F_2)$. Observe that $K_2\phi_v$ is a harmonic function on $A_2 = \bigcup_{n=2}^{\infty} (A_{n1} \cup A_{n2})$ and $v + M_2\phi_v$ is a harmonic function on Ω_2 with the same values as that of $K_2\phi_v$ on $\partial(\Omega_2 \cap A_2)$. Then we can define the harmonic function

$$u(z) = \begin{cases} K_2\phi_v(z) & (z \in A_2), \\ v(z) + M_2\phi_v(z) & (z \in \Omega_2) \end{cases}$$

in $HP(W_2; \partial W_2)$ which satisfies on Ω_2

$$T_2u = u - H_u^{\Omega_2} = v + M_2\phi_v - L_2(K_2\phi_v) = v. \quad \square$$

9. We prove $\dim\{\bar{S}_n\} = 1$, $\dim\{\bar{S}_n\} = 2$. Consider a mapping E_ν of $HP_1(W_\nu; \partial W_\nu)$ to $HP_1(\Omega_\nu; \partial\Omega_\nu)$ given by

$$E_\nu u = \frac{T_\nu u}{l(T_\nu u)} \quad (u \in HP_1(W_\nu; \partial W_\nu); \nu = 1, 2),$$

where $l(v) = -(2\pi)^{-1} \int_{\partial\Omega} (\partial/\partial|z|)v(z) |dz|$. Let u be in $\text{ex. } HP_1(W_\nu; \partial W_\nu)$ and $E_\nu u$ be represented in a form $E_\nu u = tv_1 + (1-t)v_2$ for a constant t in $(0, 1)$ and v_1, v_2 in $HP_1(\Omega_\nu; \partial\Omega_\nu)$. Then we have

$$\frac{u}{l(T_\nu u)} = tl(T_\nu^{-1}v_1) \frac{T_\nu^{-1}v_1}{l(T_\nu^{-1}v_1)} + (1-t)l(T_\nu^{-1}v_2) \frac{T_\nu^{-1}v_2}{l(T_\nu^{-1}v_2)}.$$

Therefore we have $u = T^{-1}v_1/l(T_\nu^{-1}v_1) = T^{-1}v_2/l(T_\nu^{-1}v_2)$ and hence $l(T_\nu u)E_\nu u = v_1/l(T_\nu^{-1}v_1) = v_2/l(T_\nu^{-1}v_2)$. Then $l(E_\nu u) = lv_1 = lv_2 = 1$ imply $E_\nu u = v_1 = v_2$ so that $E_\nu u$ is extreme in $HP_1(\Omega_\nu; \partial\Omega_\nu)$. Thus E_ν is a mapping of $\text{ex. } HP_1(W_\nu; \partial W_\nu)$ to

ex. $HP_1(\Omega_\nu; \partial\Omega_\nu)$. Further we can show that E_ν is bijective. Let u_1, u_2 be any functions in ex. $HP_1(W_\nu; \partial W_\nu)$ such that $E_\nu u_1 = E_\nu u_2$. Then the facts $lu_1 = lu_2 = 1$ and $u_1/l(T_\nu u_1) = u_2/l(T_\nu u_2)$ imply $u_1 = u_2$ so that E_ν is injective. Let v be any function in ex. $HP_1(\Omega_\nu; \partial\Omega_\nu)$. Then we have $E(T_\nu^{-1}v/l(T_\nu^{-1}v)) = v$. Assume $T_\nu^{-1}v/l(T_\nu^{-1}v) = tu_1 + (1-t)u_2$ for u_1, u_2 in $HP_1(W_\nu; \partial W_\nu)$ and a constant t in $(0, 1)$. Then we have $v/l(T_\nu^{-1}v) = tT_\nu u_1 + (1-t)T_\nu u_2$ and hence $v = T_\nu u_1/l(T_\nu u_1) = T_\nu u_2/l(T_\nu u_2)$. Therefore we have $T_\nu^{-1}v = u_1/l(T_\nu u_1) = u_2/l(T_\nu u_2)$. Since $lu_1 = lu_2 = 1$, we obtain $T_\nu^{-1}v/l(T_\nu^{-1}v) = u_1 = u_2$ so that $T_\nu^{-1}v/l(T_\nu^{-1}v)$ is extreme. Then E_ν is also surjective. Thus we conclude

$$(6) \quad \begin{cases} \dim\{\bar{S}_n\} = \#(\text{ex. } HP_1(\Omega_1; \partial\Omega_1)) = 1, \\ \dim\{\bar{S}_{n,j}\} = \#(\text{ex. } HP_1(\Omega_2; \partial\Omega_2)) = 2. \end{cases}$$

2. Firmly associated densities.

10. Consider a \mathcal{U} -sequence $\{\bar{Y}_n\}_1^\infty$ and a density P on Ω with $\text{supp } P \subset \cup_1^\infty \bar{Y}_n$. Solutions u of (1) on Ω are harmonic on $W = W(\{\bar{Y}_n\}) = \Omega - \cup_1^\infty \bar{Y}_n$. Then we define a mapping T_p of $PP(\Omega; \partial\Omega)$ to $HP(W; \partial W)$ by

$$T_p u = u - H_u^W \quad (u \in PP(\Omega; \partial\Omega)).$$

Similar to the mappings T_1 and T_2 given in subsection 6 the mapping T_p is order preserving, positively homogeneous, and additive. In general T_p may or may not be injective and similarly surjective. If the mapping T_p happens to be bijective, then the density P is said to be *canonically associated* with the \mathcal{U} -sequence $\{\bar{Y}_n\}$. If a density P on Ω is canonically associated with a \mathcal{U} -sequence $\{\bar{Y}_n\}$, then we have $\dim P = \dim\{\bar{Y}_n\}$. To prove this we consider a mapping E_p of $PP_1(\Omega; \partial\Omega)$ to $HP_1(W; \partial W)$ given by

$$E_p u = \frac{T_p u}{l(T_p u)} \quad (u \in PP_1(\Omega; \partial\Omega)).$$

Then similar to the mappings E_1 and E_2 given in subsection 9 the mapping E_p is a bijective mapping of ex. $PP_1(\Omega; \partial\Omega)$ to ex. $HP_1(W; \partial W)$.

11. A density Q on Ω with $P \leq Q$, $\text{supp } Q \subset \cup_1^\infty \bar{Y}_n$ may or may not be canonically associated with a \mathcal{U} -sequence $\{\bar{Y}_n\}_1^\infty$ even if a density P on Ω is canonically associated with $\{\bar{Y}_n\}$. We can actually construct both examples but we will not mention them here. If any density Q with $P \leq Q$, $\text{supp } Q \subset \cup_1^\infty \bar{Y}_n$ is always canonically associated with $\{\bar{Y}_n\}$ for a density P with $\text{supp } P \subset \cup_1^\infty \bar{Y}_n$, then we say that the density P is *firmly associated* with the \mathcal{U} -sequence $\{\bar{Y}_n\}$. Hence a firmly associated density with $\{\bar{Y}_n\}$ is a canonically associated density with $\{\bar{Y}_n\}$ and a density Q with $P \leq Q$, $\text{supp } Q \subset \cup_1^\infty \bar{Y}_n$ is also firmly associated with $\{\bar{Y}_n\}$ if a density P is firmly associated with $\{\bar{Y}_n\}$. Now we prove the following

THEOREM 2. *There always exists a density P on Ω firmly associated with an arbitrarily given \mathcal{U} -sequence.*

The proof of this theorem will be given in subsections 15–18 after establishing auxiliary results in subsections 12–14.

12. Let D be a Jordan region in Ω . We denote by P_f^D the solution of (1) on D with boundary values f on ∂D for a density P on Ω and a continuous function f on ∂D . Let $\{P_n\}_1^\infty$ be a sequence of densities P_n on Ω with

$$\lim_{n \rightarrow \infty} \left(\inf_D P_n \right) = +\infty.$$

Then we have the following

LEMMA 3. *The sequence $\{(P_n)_1^D\}_1^\infty$ converges to zero as $n \rightarrow \infty$ uniformly on every compact subset of D .*

For a proof we take any disk $U = U(p, r)$ in D with a center p and a radius r and set $c_n = \inf_{k \geq n} (\inf_D P_k)$, $w_n = (c_n)_1^U$. Since $w_n \geq w_{n+1} > 0$, the sequence $\{w_n\}_1^\infty$ converges to a nonnegative function w on U . Assume that $w > 0$ on a subset of U with positive measure. Then there exist a positive number δ and a compact subset E of $\{z \in U; w(z) > \delta\}$ such that the measure $|E|$ of E is positive. If we denote the harmonic Green's function on U by $g_U(\cdot, \cdot)$, then we have

$$1 = w_n(z) + \frac{1}{2\pi} \iint_U g_U(z, \zeta) c_n w_n(\zeta) d\xi d\eta \quad (\zeta = \xi + i\eta)$$

for any z in U . We fix a point q in U and set $\sigma = \delta \inf_E g_U(q, \cdot)$. Then we have

$$\begin{aligned} 1 &\geq w_n(q) + \frac{1}{2\pi} \iint_E g_U(q, \zeta) c_n w_n(\zeta) d\xi d\eta \\ &\geq w_n(q) + \frac{\sigma |E| c_n}{2\pi} \end{aligned}$$

so that

$$|E| \leq \frac{2\pi(1 - w_n(q))}{\sigma c_n}$$

is valid for every $n = 1, 2, \dots$. This contradicts the fact that $|E| > 0$. Therefore $w = 0$ almost everywhere on U . On the other hand, w_n is rotation free:

$$w_n(z) = w_n(|z - p| + p).$$

Further, by the maximum principle, $w_n(z_1) \leq w_n(z_2)$ if $|z_1 - p| \leq |z_2 - p|$. Then $w = 0$ everywhere on U . Thus, by the Dini theorem, $\{w_n\}$ converges to zero uniformly on every compact subset of U and hence we obtain Lemma 3 since $(P_n)_1^D \leq (c_n)_1^D \leq w_n$. \square

13. Let U, V be Jordan regions in Ω with $\bar{U} \subset V$. Let $\{P_n\}_1^\infty$ be a sequence of densities P_n on Ω such that $\text{supp } P_n \subset U$ and $\lim_n (\inf_K P_n) = +\infty$ for any compact subset K of U . Consider a subharmonic function

$$w(z; U, V) = \begin{cases} \omega(\partial V; V - \bar{U}, z) & (z \in V - \bar{U}), \\ 0 & (z \in \bar{U}) \end{cases}$$

on V . If we set $\|\phi\|_E = \sup_E |\phi|$ for a function ϕ on a subset E of Ω , then we have the following

LEMMA 4. $\lim_n \|(P_n)_1^V - w(\cdot; U, V)\|_{\bar{V}} = 0$.

Since we have $(P_n)_1^V \leq (P_n)_1^D$ for any Jordan region D in U with $\bar{D} \subset U$, the sequence $\{(P_n)_1^V\}_1^\infty$ converges to zero uniformly on every compact subset of U by Lemma 3. We take an exhaustion $\{U_m\}_1^\infty$ of U consisting of Jordan regions U_m in U with $\bar{U}_m \subset U$. Set $s_{nm} = \sup_{\bar{U}_m} (P_n)_1^V$ ($n = 1, 2, \dots; m = 1, 2, \dots$). Then $\lim_n s_{nm} = 0$, $(P_n)_1^V \leq s_{nm} + w(\cdot; U_m, V)$. Therefore we have

$$0 \leq (P_n)_1^V - w(\cdot; U, V) \leq s_{nm} + (w(\cdot; U_m, V) - w(\cdot; U, V))$$

on V . We also set $t_m = \sup_{\partial U} w(\cdot; U_m, V)$ ($m = 1, 2, \dots$). Then $\lim_m t_m = 0$ so that we obtain Lemma 4 by the inequality

$$\|(P_n)_1^V - w(\cdot; U, V)\|_{\bar{V}} \leq s_{nm} + t_m. \quad \square$$

14. Let X, Y be Jordan regions in Ω with $\bar{Y} \subset X$. Then we have the following simple but useful fact:

LEMMA 5. *For any positive number ε there exists a density P on Ω such that $\text{supp } P \subset Y$ and*

$$\|P_f^X\|_{\bar{Y}} \leq \varepsilon \|f\|_{\partial X}$$

for any f in $C(\partial X)$.

We can take a sequence $\{P_n\}_1^\infty$ of densities P_n on Ω with $\text{supp } P_n \subset Y$ and $\lim_n (\inf_K P_n) = +\infty$ for any compact subset K of Y . For example, for an exhaustion $\{Y_n\}_1^\infty$ of Y we consider nonnegative C^∞ -functions P_n with $P_n = n$ on Y_n , $P_n = 0$ on $\bar{\Omega} - Y_{n+1}$. Then by Lemma 4 we have

$$\lim_{n \rightarrow \infty} \|(P_n)_1^X - w(\cdot; Y, X)\|_{\bar{X}} = 0$$

so that $\lim_n \|(P_n)_1^X\|_{\bar{Y}} = 0$. Observe that

$$|(P_n)_f^X| \leq (P_n)_{|f|}^X \leq (P_n)_{\|f\|_{\partial X}}^X = \|f\|_{\partial X} (P_n)_1^X.$$

Then we have

$$\|(P_n)_f^X\|_{\bar{Y}} \leq \|(P_n)_1^X\|_{\bar{Y}} \|f\|_{\partial X}.$$

Therefore if we set $P = P_n$ for n with $\|(P_n)_1^X\|_{\bar{Y}} < \varepsilon$, we obtain Lemma 5. \square

15. We proceed to the proof of Theorem 2, i.e. the existence of firmly associated densities. Let $\{\bar{Y}_n\}_1^\infty$ be any \mathcal{Q} -sequence in Ω and X_n be a slightly larger Jordan region in Ω than Y_n containing \bar{Y}_n . We may assume $\bar{X}_n \cap \bar{X}_m = \emptyset$ for $n \neq m$. We fix a point z_0 in $\Omega - \bigcup_1^\infty \bar{X}_n$ and denote by F the set of nonnegative harmonic functions u on $W = W(\{\bar{Y}_n\}) = \Omega - \bigcup_1^\infty \bar{Y}_n$ with $u(z_0) = 1$. Then the Harnack inequality yields

$$\beta_n \equiv \sup_F \max_{\partial X_n} u < +\infty \quad (n = 1, 2, \dots).$$

We define a density $P = P_W$ on Ω by

$$P(z) = \begin{cases} P_n(z) & (z \in \bar{X}_n), n = 1, 2, \dots, \\ 0 & \left(z \in \bar{\Omega} - \bigcup_1^\infty \bar{X}_n\right), \end{cases}$$

where P_n is a density on Ω which satisfies Lemma 5 for $\varepsilon = (2\beta_n)^{-1}$, $Y = Y_n$, $X = X_n$. We will show that P is firmly associated with $\{\bar{Y}_n\}$.

16. First we prove that the mapping T_p of $PP(\Omega; \partial\Omega)$ to $HP(W; \partial W)$ is *injective*. Let u_1, u_2 be any functions in $PP(\Omega; \partial\Omega)$. We remark that u_1, u_2 are harmonic on W . Then u_1, u_2 satisfy $u_1 \leq \beta_n u_1(z_0), u_2 \leq \beta_n u_2(z_0)$ on ∂X_n . Therefore by Lemma 5 we have $u_1 \leq u_1(z_0)/2, u_2 \leq u_2(z_0)/2$ on $\bigcup_1^\infty \bar{Y}_n$ so that u_1, u_2 are bounded on $\partial W = \partial\Omega \cup (\bigcup_1^\infty \partial \bar{Y}_n)$. Assume $T_p u_1 = T_p u_2$. Since $u_1 - u_2 = H_{u_1 - u_2}^W$ on W , $u_1 - u_2$ is bounded on W and hence on Ω . Further $u_1 - u_2$ vanishes on $\partial\Omega$. Then we have $u_1 - u_2 \equiv 0$.

17. Next we prove that T_p is *surjective*. We set

$$F_X = \bigcup_1^\infty \partial X_n, \quad \Gamma_Y = \bigcup_1^\infty \partial Y_n.$$

Consider a bounded operator K_p of $C(F_X)$ to $C(\Gamma_Y)$ defined by

$$K_p \phi(z) = P_\phi^{X_n}(z) \quad (\phi \in C(F_X), z \in \partial Y_n; n = 1, 2, \dots).$$

Also consider a bounded operator L_p of $CB(\Gamma_Y)$ to $CB(F_X)$ defined by

$$L_p \psi(z) = H_\psi^W(z) \quad (\psi \in CB(\Gamma_Y), z \in F_X),$$

where we set $\psi = 0$ on $\partial\Omega$. Let v be any function in $HP(W; \partial W)$. Then we can construct a function u in $PP(\Omega; \partial\Omega)$ with $T_p u = v$ by using the bounded operator $M_p = L_p \circ K_p$ of a subspace of $C(F_X)$ to $CB(F_X)$. Since v is dominated by the constant $\beta_n v(z_0)$ on ∂X_n , we have $K_p^m v \leq v(z_0)/2$ on Γ_Y by Lemma 5. Then a harmonic function $M_p v$ is dominated by $v(z_0)/2^m$ ($m = 1, 2, \dots$). Therefore there exists a nonnegative function

$$\phi_v = \sum_{m=0}^\infty M_p^m v$$

on F_X with $(I - M_p)\phi_v = v$, where I is the identity operator of $C(F_X)$. Observe that $K_p \phi_v$ is a solution of (1) on $\bigcup_1^\infty X_n$ and $v + M_p \phi_v$ is a solution of (1) on W with the same values as that of $K_p \phi_v$ on $\partial(W \cap (\bigcup_1^\infty X_n)) = F_X \cup \Gamma_Y$. Then we can define the function

$$u(z) = \begin{cases} K_p \phi_v(z) & (z \in \bigcup_1^\infty X_n), \\ v(z) + M_p \phi_v(z) & (z \in W) \end{cases}$$

in $PP(\Omega; \partial\Omega)$ which satisfies

$$T_p u = u - H_u^W = v + M_p \phi_v - L_p(K_p \phi_v) = v$$

on W .

18. In the preceding subsections we have shown that P is canonically associated with $\{\bar{Y}_n\}$. We turn to the proof that P is firmly associated with $\{\bar{Y}_n\}$. Take any density Q on Ω with $\text{supp } Q \subset \bigcup_1^\infty \bar{Y}_n$ and $P \leq Q$. Consider densities P_n, Q_n ($n = 1, 2, \dots$) on Ω defined by

$$P_n(z) = \begin{cases} P(z) & (z \in \bar{X}_n), \\ 0 & (z \in \bar{\Omega} - \bar{X}_n), \end{cases} \quad Q_n(z) = \begin{cases} Q(z) & (z \in \bar{X}_n), \\ 0 & (z \in \bar{\Omega} - \bar{X}_n). \end{cases}$$

Then $\text{supp } P_n \subset \bar{Y}_n$, $\text{supp } Q_n \subset \bar{Y}_n$ and $P_n \leq Q_n$. Similar to the density P_n , the density Q_n satisfies Lemma 5 for $\varepsilon = (2\beta_n)^{-1}$, $X = X_n$, $Y = Y_n$ since $(Q_n)_1^{X_n} \leq (P_n)_1^{X_n}$. Then Q is canonically associated with $\{\bar{Y}_n\}$ and hence P is firmly associated with $\{\bar{Y}_n\}$. \square

3. The proof of the main theorem.

19. Let $\{\bar{S}_n\}_1^\infty$, $\{\bar{S}_{nj}\}_{j=1,2; n \geq 1}$ be \mathcal{U} -sequences given in subsection 6. We can take firmly associated densities P_1 and Q on Ω with $\{\bar{S}_n\}$ and $\{\bar{S}_{nj}\}$, respectively. We set $P = P_1 + Q$. Then $\text{supp } P \subset (\cup_1^\infty \bar{S}_n) \cup (\cup_1^\infty (\bar{S}_{n1} \cup \bar{S}_{n2})) = \cup_1^\infty \bar{S}_n$, and $P_1 \leq P$. Therefore P is also canonically associated with $\{\bar{S}_n\}$. Thus we deduce

$$\dim Q = \dim \{\bar{S}_{nj}\} = 2, \quad \dim P = \dim \{\bar{S}_n\} = 1,$$

although $Q \leq P$. \square

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